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Design of interactive systems—a formal approach

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Decision Support Systems (DSSs) are utilized to support a users decision process. One generally required characteristic of a DSS is that it be an interactive system. Generally the degree of interaction between the human and the system is such that one can view the information processing activity as being performed by the human-computer information processor. Although DSSs are fairly commonly used, there has been very little work done to develop a formal basis for the design of such systems which take into account the interactive nature of problem solving. In this paper we propose a formal model for analysing the human–machine information processor. The model takes into account cost of performing information-gathering actions, communication costs and time constraints. We illustrate the application of the model within the domain of categorization. A special case of the categorization problem called the "only-correct-guesses-count" problem is defined and analyzed within the context of the model.

1. Introduction

Decision aids such as Decision Support Systems (DSSs) are designed to function in an interactive mode with the user. The interaction between the human and the system could take several forms. A situation in which the user types in a request, the system responds and the user leaves with the results, would correspond to the no interaction case. In most cases before the final results are obtained, there is an exchange of information between the human and the computer. The interaction could be user directed, i.e. the user obtains some results from the system and queries it further for more information, requests alternative solutions or directs the system's problem-solving approach. On the other hand there are systems, for example expert systems, in which the interaction is largely directed by the system. One could also have a combination of user directed and system-directed interaction. Systems such as Symbiotic Decision Support Systems (SDSS) (Manheim, 1989) and Active DSS (Mili, 1989) can be considered to function along these lines.

In addition to the nature of interaction, a critical factor in the design of a DSS is the time and cost involved in performing relevant information-gathering actions while

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solving the problem. Most problems have to be solved within a finite time constraint. For example, placing a bid in a sealed-bid action, requires the bidder to obtain information and place a bid before the deadline specified by the auctioneers. Solving a decision problem typically implies a net positive payoff to the decision maker. Given the speed and storage capabilities of the computer, one rationale for its use in decision making is to increase the number of information-gathering actions performed before making a decision. However, there are problems (e.g. NP complete problems) which in spite of the speed of the computer could take a phenomenal amount of time to solve completely. Often in such cases rather than obtain precise solutions, heuristics are utilized to obtain good solutions, with the precision of the solution dependent on factors such as time, assumptions made and information required to execute the algorithm. In such cases it is necessary to balance the cost vs. benefits of the solution approach. Thus, in utilizing the information-gathering action, i.e. an algorithm, in the process of solving the decision problem, one needs to consider its cost and execution time.

Given the interactive nature of problem solving, the question arises can one develop a formal model of decision making, incorporating the idea that problem solving has to be viewed in a human–machine context, while at the same time considering cost of information gathering and time constraints. In this paper we propose such a model. The underlying concepts of the model draws on decision theory. A special case of the categorization problem called the "only-correct-guesses-count" problem is analyzed using the model.

The paper is organized as follows: the features of the model are discussed in Section 2 of this paper. In section 3, the categorization problem is defined within the context of the decision-making model and problem-solving strategies for the only correct guesses count problem are defined. The conclusions and extensions to the work are discussed in Section 4. The appendix contains a formal analysis of the only-correct-guesses-count problem and the nature of the optimal solution for the problem.

2. A model of decision making

The model proposed here is an extension of the model presented in Moore and Whinston (1986, 1987). Various characteristics of the extended model are considered in Jacob, Moore and Whinston (1989). Here we relate the approach to interactive problem-solving and discuss its application within the context of categorization. In defining the decision problem we consider several factors:

1. Given a state space the goal of the problem-solving activity is seen as determining the "true state".
2. In general one knows something about the problem and this is reflected in the probability density function which is assumed to be known.
3. If the true state were known the set of possible final decisions corresponding to it are known.
4. One performs information-gathering actions to determine the true state. The information-gathering actions could include such activities as execution of an algorithm.

5. Given the fact that there are two information processors, the information-gathering action could belong to either the human or the computer.
6. Performing an information-gathering action by either the human or computer is going to incur some cost, as well as take time.
7. The problem needs to be solved within a deadline or in other words there is time pressure to complete the problem.

The decision problem is formally defined by nine elements:

$$D = \langle X, \phi, D, \omega^*, A, \{M_a \mid a \in A\}, c, t, T \rangle$$

where

X is the set of possible (mutually exclusive) states. The generic notation "x" is used to denote elements of X.

ϕ $X \rightarrow [0, 1]$ is the probability density function. ϕ defines the probability distribution function π: $P(X) \rightarrow [0, 1]$ by:

$$\pi(Y) = \sum_{x \in Y} \phi(x) \quad \text{for} \quad Y \subseteq X,$$

where "$P(X)$" denotes the power set of x.

D is the set of available (final) decisions.

ω^*: $X \times D \rightarrow R$ is the payoff function.

A is the set of "initial" (information-gathering) actions, or experiments, available.

M_a is the information structure associated wtih action $a \in A$. (Each M_a is a partition of x, as will be explained in more detail below.)

c: $A \rightarrow R_+$ is the cost function; $c(a)$ is the cost of utilizing action $a \in A$.

t: $A \rightarrow R_+$ is the time function; $t(a)$ is the time taken to perform action $a \in A$.

T is the total time available in which to perform the information-gathering actions. (T could represent the time beyond which the state changes. It could also represent the time after which the payoff is negligible or zero.)

Within the above context we also assume that:

1. The state space, X, the set of decisions, D and the action set, A are all finite.
2. The true state $\hat{x} \in X$ does not change while the decision problem is being solved. Alternatively one could view T as the time beyond which the state changes.
3. Given there is the human and the computer involved the action set A is made up of information gathering, and communication actions from the two information processors. A, therefore is defined as:

$$A = A_1 \cup A_2$$

where,

$$A_1 = A_1^e \cup A_1^c$$

$$A_2 = A_2^e \cup A_2^c$$

The subscript 1 refers to the human information processor and the subscript 2 refers to the computer. The superscript e denotes the information-gathering or experimental action set and the superscript c denotes the communication action set.

The decision problem solving activity is seen as performing information-gathering actions which partition the state space. The definition below formalizes the concept of an information structure on a set B.

DEFINITION 1

Let $B \subseteq X$ be non-empty. A family of subsets of X, $\boldsymbol{B}$, is an information structure on B iff:

i. $(\forall B' \in \boldsymbol{B}): B' \neq \phi$.
ii. $\boldsymbol{B}$ is a partition of B (That is, the sets in $\boldsymbol{B}$ are pairwise disjoint, and their union equals B).

Given the above definition of an information structure, we can now describe the result of performing an information-gathering action. Associated with each $a \in A_s^e$, $s = 1, 2$ is a set of information signals, Y_a, and a function $\eta_a : X \to Y_a$. Each Y_a is assumed to contain a finite number, $n(a)$, of different signals, so that, without loss of generality Y_a can be written as

$$Y_a = \{1, 2, \ldots, n(a)\}.$$

It is also assumed that $a = 0$ is the null information action and $n(0) = 1$.

For a given $x \in X$, there is a single signal receivable from each of the n information signal sets. Thus the information is viewed to be obtained deterministicaly (i.e. noiseless information). Thus, if the information-gathering action $a \in A_s^e$, $s = 1, 2$, is performed and the signal $y \in Y_a$ is received, it is known that the true state $\hat{x}$, is an element of the set M_{ay} defined by:

$$M_{ay} = \{x \in X \mid \eta_a(x) = y\} = \eta_a^{-1}(\{y\}) \quad \text{for} \quad a = 0, 1, \ldots, n, y = 1, \ldots, n(a)$$

Notice that M_a defined by

$$M_a = \{M_{a1}, \ldots, M_{a,n(a)}\} \quad \text{for} \quad a = 0, 1, \ldots, n.$$

will be a partition of X. After action a is performed, one will know to which of M_{ay} the true state belongs. It should also be clear that, $\forall a \in A_1^e \cup A_1^e$, M_a is an information structure on X (by Definition 1). M_a is referred to as the information structure associated with (or induced by) a. A communication action results in the communication of some $B \subseteq X$ from one processor to the other. However, as far as the processor which is sending the message is concerned the communication action does not partition its state space as the information-gathering does, therefore, one can view for $a \in A_1^c \cup A_2^c$, $M_a = \{X\}$.

DEFINITION 2

Let $B \subseteq X$ be non-empty, and let $a \in A$. The information structure induced on B by a, $\iota(B, a)$, is defined as:

$$\iota(B, a) = \begin{cases} \{B \cap M_{a1}, & B \cap M_{a2}, \ldots, B \cap M_{a,n(a)}\} \setminus \{\phi\} \qquad \text{if } a \in A_1^e \cup A_2^e \\ \{B \cap M_a\} & \text{if } a \in A_1^c \cup A_2^c \end{cases}$$

DEFINITION 3

Let $B \subseteq X$ be non-empty, and let $\boldsymbol{B}$ be an information structure on B. An action function on $\boldsymbol{B}$ is a function $\alpha : \boldsymbol{B} \to A$.

In general, the goal of performing actions is to refine an information structure or in other words further partition a set. The next definition formalizes the notion of refinement.

DEFINITION 4

Let $B \subseteq X$ be non-empty, let $\boldsymbol{B} = \{B_1, \ldots, B_k\}$ be an information structure on $\boldsymbol{B}$, and let $\alpha : \boldsymbol{B} \rightarrow A$ be an action function on $\boldsymbol{B}$. The refinement of $\boldsymbol{B}$ by α, $R(\boldsymbol{B}, \alpha)$, is defined by:

$$R(\boldsymbol{B}, \alpha) = \bigcup_{j=1}^{k} \iota[B_j, \alpha(B_j)].$$

DEFINITION 5

Let $B \subseteq X$ be non-empty, and let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be information structures on B. $\boldsymbol{B}_1$ is said to be as fine as $\boldsymbol{B}_2$ (or that $\boldsymbol{B}_1$ is a refinement of $\boldsymbol{B}_2$), and written as $\boldsymbol{B}_1 \geq \boldsymbol{B}_2$, iff:

$$(\forall B' \in \boldsymbol{B}_1)(\exists B'' \in \boldsymbol{B}_2) : B' \subseteq B''.$$

Note that if B is a non-empty subset of X, $\boldsymbol{B}$ is an information structure on B, and α is an action function on $\boldsymbol{B}$, then $R(\boldsymbol{B}, \alpha)$ is (an information structure on B and is) a refinement of $\boldsymbol{B}$.

So far we have considered the issue of partitioning the state space using information-gathering actions. The issue arises if there is a time constraint how many information-gathering actions can be taken before a decision has to be made. The next definition specifies the maximum number of actions that can be taken given a time constraint.

DEFINITION 6

Let $a^* \in A'$, where A' indicates the set of experimental actions other than the null information action, be such that

$$\forall a \in A'$$
$$t(a) \geq t(a^*).$$

If we then define

$$r = \left\lfloor \frac{T}{t(a^*)} \right\rfloor$$

r is the maximum number of information-gathering actions possible in a strategy.

It is obvious that the number of actual information-gathering actions performed along any path in a strategy will generally be some $r' < r$. However, since both the cost and time taken to perform a null information-gathering action are assumed to be zero (i.e. $c(0) = 0$ and $t(0) = 0$), therefore, it is assumed that exactly r actions are performed along any given path, with $r - r'$; of them being the null information-gathering action.

A two processor strategy for D, σ, is defined as:

$$\sigma = \langle [(\boldsymbol{B}_1, \alpha_1)], [(\boldsymbol{B}_2^1, \alpha_2^1), (\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1), (\boldsymbol{B}_r^2, \alpha_r^2)], [(\boldsymbol{B}_{r+1}^1, \delta)] \rangle$$

satisfying:

1. $\boldsymbol{B}_1 = \{X\}$
2. a. $\alpha_j^s : \boldsymbol{B}_j^s \to A_s$ for $j = 2, \ldots, r; s = 1, 2$
   b. $\alpha_1^1 : \boldsymbol{B}_1^1 \to A_1$
   c. $\boldsymbol{B}_{j+1}^s = R(\boldsymbol{B}_j^s, \alpha_j)$ for $j = 1, 2, \ldots, r; s = 1, 2.$
3. $\delta : \boldsymbol{B}_{r+1}^1 \to D.$

A strategy is viewed as being composed of two parts

i. the *information-gathering strategy*:

$$\alpha = \langle [(\boldsymbol{B}_1, \alpha_1)], [(\boldsymbol{B}_2^1, \alpha_2^1), (\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1), (\boldsymbol{B}_r^2, \alpha_r^2)] \rangle,$$

ii. the *decision strategy*, $(\boldsymbol{B}_{r+1}', \delta)$.

Within this context we make the following assumptions about the behaviour of the human and the computer:

1. The decision problem originates with the human, who initiates the problem solving and the final decision is made by the human.
2. Each processor can perform only one action at a given time, i.e. communication or information-gathering. Note that communication is used to denote the combined operation of sending and receiving information.
3. The computer is assumed to have the capabilities of a DSS. The communication actions performed by the humans are, therefore, non-programming actions (i.e., actions not involving writing programs). This also implies that the computer can perform multiple information-gathering actions if necessary before responding to the human.

In creating a dual processor strategy we are in effect attempting to design an interactive system, deeping in mind that the two processors have very different capabilities and that decision-making has to be viewed as a combined effort between the two processors. Given the context of decisionmaking, it is natural to use an expected payoff measure of determining, "how good an interactive information-gathering strategy really is." This will also provide a comparative measure for the design of interactive systems. The next definitions allow us to define the expected net payoff of a strategy.

DEFINITION 7

If

$$\sigma = \langle [(\boldsymbol{B}_1, \alpha_1)], [(\boldsymbol{B}_2^1, \alpha_2^1), (\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1), (\boldsymbol{B}_r^2, \alpha_r^2)], [(\boldsymbol{B}_{r+1}^1, \delta)] \rangle$$

is a feasible strategy for D, and $q \in \{1, \ldots, r)$, the sequence $\langle \beta_j^s(B) \rangle$ $(j = 1, \ldots, q)$ for each $B \in \boldsymbol{B}_q^s$ (as well as for $B \in \boldsymbol{B}_{r+1}'$ and $j = 1, \ldots, r$) is defined by:

$$\beta_j^s(B) = \text{that } B' \in B_j \text{ such that } B \subseteq B' \qquad s = 1, 2.$$

$\beta_j^s(B)$ is referred to as the predecessor of B at j, the superscript indicates whether the predecessor results from an action done by the human or the DSS.

DEFINITION 8

Let

$$\sigma = \langle [(\boldsymbol{B}_1, \alpha_1)], [(\boldsymbol{B}_2^1, \alpha_2^1), (\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1), (\boldsymbol{B}_r^2, \alpha_r^2)], [(\boldsymbol{B}_{r+1}^1, \delta)] \rangle$$

be a feasible strategy for D, and let $q \in \{2, \ldots, r\}$. For each $B \in \boldsymbol{B}_q^s$, $a_q^s(B)$ is defined as the sequence (of length $q-1$) of actions taken by the strategy σ along the path that yields B. Thus:

$$a_q^s(B) = \langle a^s(1, B), \ldots, a^s(q-1, B) \rangle, \qquad s = 1, 2$$

where "$a^s(j, B)$" denotes the action taken at step j ($j = 1, \ldots, q-1$) along the path that yields B, i.e.,

$$a^s(j, B) = \alpha_j^s[\beta_j^s(B)] \quad \text{for} \quad j = 1, \ldots, q-1.$$

Note that the above definition also holds for $B \in \boldsymbol{B}_{r+1}^1$.

In a given realization of the decision problem, the application of a strategy, will result in the determination that $\hat{x}$, the true state, is an element of some $B \in B_{r+1}$. The cost of determining that $\hat{x} \in B$ will be the sum of the costs of all the actions taken along the path yielding B, and will therefore be given by:

$$C(B) = \sum_{j=1}^{r} c[a^1(j, B)] + \sum_{j=1}^{r} c[a^2(j, B)].$$

The expected informational cost of strategy σ is therefore given by:

$$\Gamma(\sigma) = \sum_{B \in \boldsymbol{B}_{r+1}^1} \pi(B) C(B).$$

Given a strategy for D, σ, the gross expected payoff for σ is denoted by "$\Omega(\sigma)$" is given by

$$\Omega(\sigma) = \sum_{B \in \boldsymbol{B}_{r+1}^1} \sum_{x \in B} \phi(x) \omega^*[x, \delta(B)].$$

The expected net payoff of executing a particular strategy is given by:

$$\Omega^*(\sigma) = \Omega(\sigma) - \Gamma(\sigma)$$

The goal, therefore, is to determine the best strategy for solving the problem, or in other words to find the strategy which maximizes the expected net payoff.

Given that the computer has the capabilities of a DSS, from the perspective of designing an interactive system, the communication action points are assumed to be preset. Although one might view this as a restrictive assumption, it is not completely unfounded. Clearly the design of an interactive system assumes certain capabilities on the part of the two processors. The system can perform certain information-gathering actions such as execution of models or accessing data from a data base, at the same time the system may need information obtained by the human as a result of performing information-gathering actions on the environment. The system would, therefore, query the user for the required information whenever necessary. Although one may argue that the user could provide a substantial amount of information at the outset of the interaction, this will be inefficient as the human would perform information-gathering actions and generate information which the system might not utilize. Thus, in designing systems which take an integrated approach to decision problem solving one needs to take into account when the two processors should communicate.

DEFINITION 9

The set of communication points, $Q \subset \{1, \ldots, r\}$ is defined by:

$$Q = \{j \in \{1, \ldots, r\} \mid (\exists s \in \{1, 2\}, B \in \boldsymbol{B}_j^s, \alpha_j^s(B) \in A_s^c\}.$$

If Q is not empty, we shall denote it by:

$$Q = \{q_1, q_2, \ldots, q_p\}$$

where $1 \leq q_1 < q_2 < \cdots < q_p \leq r$.

Given the human–computer structure of the problem-solving endeavour, the following features of the interaction is assumed:

1. The first communication is performed by the human and until that occurs the computer does not perform any information-gathering actions.
2. At the instant the human (computer) communicates to the computer (the human), it (he/she) is not performing any actions pertaining to the problem at hand. In other words the human cannot instruct the computer about the problem while it is in the process of performing an information gathering action or when it is providing information to the human. Conversely if the system provides the human with information while the human is performing an information-gathering action, he/she will address it only after he/she has completed the action which was being performed.
3. It is assumed that after the computer responds to the users request, it does not perform further information-gathering actions until requested to do so again by the human.
4. A communication action does not partition the communicator's information structure, but partitions the information structure of the processor to whom the information was communicated.

DEFINITION 10

Let $\boldsymbol{B}_e = \{B_1^e, B_2^e, \ldots, B_{n_e}^e\}$, and $\boldsymbol{B}_0 = \{B_1^0, B_2^0, \ldots, B_{n_0}^0\}$. Then $\cap(\boldsymbol{B}_e, \boldsymbol{B}_0)$ is defined as

$$\begin{aligned}\cap(\boldsymbol{B}_e, \boldsymbol{B}_0) = \{&B_1^e \cap B_1^0, B_1^e \cap B_2^0, \ldots, B_1^e \cap B_{n_0}^0,\\ &B_2^e \cap B_1^0, \ldots, B_2^e \cap B_{n_0}^0, \ldots,\\ &B_{n_e}^e \cap B_1^0, \ldots, B_{n_e}^e \cap B_{n_0}^0\} \setminus \{\phi\}.\end{aligned}$$

Based on the characteristics discussed earlier we will define a feasible dual processor strategy.

DEFINITION 11

A dual processor strategy.

$$\sigma = \langle [(\boldsymbol{B}_1, \alpha_1^1)], [(\boldsymbol{B}_2^1, \alpha_2^1)(\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1)(\boldsymbol{B}_r^2, \alpha_r^2)], [(\boldsymbol{B}_{r+1}^1, \delta)] \rangle$$

is called *viable* iff either:

A) $Q = \phi$, in which case

$$\boldsymbol{B}_j^2 = \{X\} \quad \text{and} \quad \alpha_j^2 : \boldsymbol{B}_j^2 \to \{0\} \quad \text{for} \quad j = 2, 3, \ldots, r.$$

Or

B) There exists a positive integer k such that

$$\#Q = p = 2k$$

where $\#Q$ indicates the number of elements of Q

and

1) For each $i \in \{1, \ldots, p\}$ one has:
   a) If i is odd then $\alpha_{q_i}^1 : \boldsymbol{B}_{q_i}^1 \to A_1^c$, $\boldsymbol{B}_{q_i+1}^1 = \boldsymbol{B}_{q_i}^1$, $\alpha_{q_i}^2 : \boldsymbol{B}_{q_i}^2 \to \{0\}$ and $\boldsymbol{B}_{q_i+1}^2 = \cap(\boldsymbol{B}_{q_i}^1, \boldsymbol{B}_{q_i}^2)$.
   b) If i is even then

$$\alpha_{q_i}^1 : \boldsymbol{B}_{q_i}^1 \to \{0\}, \qquad \boldsymbol{B}_{q_i+1}^1 = \cap(\boldsymbol{B}_{q_i}^1, \boldsymbol{B}_{q_i}^2).$$
$$\alpha_{q_i}^2 : B_{q_i}^2 \to A_2^c \quad \text{and} \quad \boldsymbol{B}_{q_i+1}^2 = \boldsymbol{B}_{q_i}^2.$$

2)
   a) If $q_1 > 1$ then $\boldsymbol{B}_j^2 = \{X\}$ and

$$\alpha_j^2 : \boldsymbol{B}_j^2 \to \{0\} \quad \text{for} \quad j = 2, \ldots, q_1.$$

   b) If i is even, $i < p$ and $q_i + 1 < q_{i+1}$, then

$$\boldsymbol{B}_j^2 = \boldsymbol{B}_{q_i}^2 \quad \text{and} \quad \alpha_j^2 : \boldsymbol{B}_j^2 \to \{0\} \quad \text{for} \quad j = q_i + 1, \ldots, q_{i+1}$$

   c) If $p < r$ then $\boldsymbol{B}_j^2 = \boldsymbol{B}_{q_p}^2$ and

$$\alpha_j^2 : \boldsymbol{B}_j^2 \to \{0\} \quad \text{for} \quad j = q_p + 1, \ldots, r$$

3) For each $j \in \{1, \ldots, r\} \setminus Q$, one has,

$$\alpha_j^s : \boldsymbol{B}_j^s \to A_s^e \quad \text{and} \quad \boldsymbol{B}_{j+1}^s = R(\boldsymbol{B}_j^s, \alpha_j^s) \quad \text{for} \quad s = 1, 2.$$

The above definition formalizes the assumptions made earlier about the form of the interaction between the human and the computer. Since there is a time constraint which has to be satisfied before a strategy can be considered feasible, the next definition establishes the constraining time between a communication action by the human and the response of the computer.

DEFINITION 12

For each $h \in \left\{1, \ldots, \frac{p}{2}\right\}$ and each $B \in \boldsymbol{B}_{r+1}^1$, $\tau_h(B)$ is defined as

$$\tau_h(B) = \max\left\{\sum_{j=q_{2h-1}+1}^{q_{2h}-1} t(a^1(j, B)), \sum_{j=q_{2h-1}+1}^{q_{2h}-1} t(a^2(j, B))\right\}$$
$$+ t(a^1(q_{2h-1}, B)) + t(a^2(q_{2h}, B)) + \sum_{j=q_{2h}+1}^{q_{2h+1}-1} t(a^1(j, B)),$$

where $q_{p+1} = r + 1$, and

$$\sum_{j=q_{2h}+1}^{q_{2h+1}-1} t(a^1(j, B)) = 0$$

if $q_{2h} + 1 = q_{2h+1}$.

A feasible strategy can now be defined in the dual processor case.

DEFINITION 13

A dual processor strategy

$$\sigma = \langle [(\boldsymbol{B}_1, \alpha_1^1)], [(\boldsymbol{B}_2^1, \alpha_2^1)(\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1)(\boldsymbol{B}_r^2, \alpha_r^2)], [(\boldsymbol{B}_{r+1}^1, \delta)] \rangle$$

is *feasible* iff,

a) σ is viable, and

b) $\forall B \in \boldsymbol{B}_{r+1}^1$, $\sum_{j=1}^{q_1-1} t(a^1(j, B))) + \sum_{h=1}^{p/2} \tau_h(B) \leq T$

A feasible strategy can be classified into one of four categories namely, independent, sequential, concurrent or mixed strategies. A strategy is called independent if only one processor performs information-gathering actions, thus within our context, either the human solves the problem by him/herself or the system solves the problem by itself. A strategy is called sequential if at any given time only one processor is performing information-gathering actions. Thus, during the problem-solving process, the two processors exchange information periodically and the exchanged information forms the basis for a processor to continue with performing information-gathering actions. A concurrent strategy has only the human performing a communication action at the start of the strategy after which both processors perform information-gathering actions till the last action where the computer communicates back to the human. Finally any strategy which does not belong to either one of the above categories are called mixed. We formally state this as follows:

DEFINITION 14

A feasible strategy σ is called:

1) *independent* if it satisfies condition A of the viability definition (Definition 11), or $Q = (1, r)$ and $\alpha_j^1 : \boldsymbol{B}_j^1 \to 0$ for $j = 2, \ldots, r-1$, $\alpha_j^2 : \boldsymbol{B}_j^2 \to A_2^c$ for $j = 2, \ldots, r-1$, $\alpha_1^1 : \boldsymbol{B}_1^1 \to A_1^c$, and $\alpha_r^2 : \boldsymbol{B}_r^2 \to A_2^c$
2) *sequential* if $Q \neq \phi$ and for $1 \leq i \leq p-1$ and i odd then,

$$\alpha_j^1 : \boldsymbol{B}_j^1 \to \{0\} \quad \text{for} \quad j = q_i + 1, \ldots, q_{i+1},$$

3) *concurrent* iff $Q = \{1, r\}$ and for $j = 2, 3, \ldots, r-1$

$$\alpha_j^1 : \boldsymbol{B}_j^1 \to A_1^c$$

and

$$\alpha_j^2 : \boldsymbol{B}_j^2 \to A_2^c$$

All other strategies are referred to as mixed strategies. Notice that a mixed strategy may be mixture of concurrent and sequential strategies, in that for i odd, there may be situations in which between q_i and q_{i+1}, both the human and the computer are performing information-gathering actions. However, in other situations only the computer may be performing information-gathering actions while the human waits for the results.

The set of feasible strategies under the dual processor framework would include independent, sequential, concurrent and mixed strategies. From this set of feasible strategies one can define a subset of strategies called efficient strategies. A feasible information-gathering strategy to be efficient has to satisfy the following conditions:

1. Given a $B \subseteq X$, if an experimental action is performed to partition it, then the information structure induced on B by the action should have at least two elements. Formally, for each $j \in \{1, 2, \ldots, r\} \setminus Q$ and $s \in \{1, 2\}$ and each $B \in \mathbf{B}_j^s$; if $\alpha_j^s(B) = \hat{a} \neq 0$ then $\#\iota(B, \hat{a}) \geq 2$.

   Clearly if this condition does not hold we have incurred a cost without any gain, and as such any strategy for which this does not hold will be strictly dominated.
2. No further information gathering actions should be taken on B, if the best decision one can take for all the elements of the information structure generated as a result of performing an information-gathering action on B is the same. Formally:

   If B^{A^c} is the finest information structure possible, i.e. if all the information-gathering actions are taken the information structure resulting from it is B^{A^c}. Let $D^*(B)$ be the set of best decisions for B, i.e.

$$D^*(B) = \left\{ d \in D \;\middle|\; \sum_{x \in B} \phi(x \mid B)\omega(x, d) = \max_{d \in D} \sum_{x \in B} \phi(x \mid B)\omega(x, d) \right\}$$

   Let $\mathbf{B}^A(d) = \{B \in B^{A^c} \mid d \in D^*(B)\}$ and $X_d = \bigcup_{B \in \mathbf{B}^A(d)} B$.
   Then for each $j \in \{1, 2, \ldots, r\}$ and each $B \in \mathbf{B}_j^s$; if for some $d \in D$, $B \subseteq X_d$, then

   a If $s = 1$, or
   b. $s = 2$, and for some $h \in \{1, \ldots, p/2\}$ $2h < j \leq 2h + 1$, we have $\alpha_q^1(B) = 0$, for $q = j, j+1, \ldots, r$.

3. If the human performs a communication action, one expects the computer to further refine the information structure. That is, if q_i and $q_{i+1} \in Q$ are such that

$$\alpha_{q_i}^1 : \mathbf{B}_{q_i}^1 \to A_1^c$$

   and

$$\alpha_{q_{i+1}}^2 : \mathbf{B}_{q_{i+1}}^2 \to A_2^c$$

   then

$$\mathbf{B}_{q_{i+1}+1}^1 > \mathbf{B}_{q_{i+1}}^1 \quad \text{and} \quad \mathbf{B}_{q_{i+1}+1}^1 \geq \mathbf{B}_{q_{i+1}}^2$$

If this condition is not met by a strategy, then we have a strategy for which the cost of communication is not warranted by the results. Additionally this strategy would be dominated by another strategy, which is identical to the first except for the communication.

4. For each $B \in \boldsymbol{B}^1_{r+1}$, $\delta(B) \in D^*(B)$, i.e. we take the best decision given B.

The set of efficient dual processor strategies can be defined as

$$\Sigma(D) = \Sigma^I(D) \cup \Sigma^C(D) \cup \Sigma^M(D) \cup \Sigma^S(D)$$

where

$\Sigma^I(D)$ = set of all efficient independent strategies for D.

$\Sigma^C(D)$ = set of all efficient concurrent strategies for D.

$\Sigma^M(D)$ = set of all efficient mixed strategies for D.

$\Sigma^S(D)$ = set of all efficient sequential strategies for D.

The optimal strategy would be a member of the set $\Sigma(D)$. In general one needs to consider strategies satisfying the efficiency conditions for further analysis. In the next section we discuss the application of the model to the categorization model.

3. The categorization problem

The categorization problem deals with the issue of classifying the current state as one of n categories. In many problem domains classification forms the core problem, whose solution paves the way for further analysis of the problem, medical diagnosis, fault diagnosis, chemical analysis of an unknown substance and forecasting are examples of areas in which solving the classification or categorization problem is the first step towards a complete solution of the problem.

Since categorization problems are encountered in practically every field, classification has been studied closely by researchers in Artificial Intelligence (Chandrasekaran, 1983; Clancey, 1984). The prevalence of categorization problems has resulted in a number of expert systems, in diverse problem domains, being constructed to solve classification problems. Table 1 lists a few of these systems.

Here we consider the problem within the context of human–computer information processing framework. In addition to the assumptions made in defining the model in Section 2, the categorization problem is defined by the following assumptions.

1) The final decision set can be written in the form:

$$D = \{0, 1, 2, \ldots, \rho\}, \quad \text{where } \rho \geq 1 \text{ is a positive integer}$$

2) There exists a partition of X, $\{X_0, X_1, \ldots, X_\rho\}$ such that ω takes the form

$$\omega(x, d) = \begin{cases} \bar{\omega} > 0 & \text{if } x \in X_d \\ 0 & \text{otherwise} \end{cases}$$

where,

$\{X_0, X_1, \ldots, X_\rho\}$ represents an exhaustive set of categories to which the true state $\hat{x}$ may belong.

Thus in the formulation, the payoff is a constant (constant over categories) $\bar{\omega}$ if $\hat{x}$ is categorized correctly otherwise the payoff is zero.

TABLE 1
Expert systems for classification problems

Expert system	Problem domain
Dendral (Buchanan & Feiganbaum, 1978)	Determines molecular structure of unknown compound from mass spectral and nuclear magnetic response data
CRIB (Hartely, 1984)	Location of computer hardware and software faults
DART (Bennet & Hollander, 1981; Genesereth, 1984)	Diagnosing faults in computer hardware systems
DIPMETER ADVISOR (Davis *et al.*, 1981)	Determines subsurface geological structure
PROSPECTOR (Gashnig, 1982)	Determines likelihood of finding mineral ore deposits
ABEL (Patil *et al.*, 1981)	Diagnosis of acid-base and electrolyte disorders in medicine
CASNET (Szolovitz & Parker, 1978)	Diagnoses disease status related to glaucoma
INTERNIST (Pople, 1982)	Diagnoses in general internal medicine
MYCIN (Shortliffe, 1976)	Diagnoses and suggests therapy for patients with bacteremia, meningitis and cystitis infections

The only correct guesses count problem is defined by adding the following assumptions to the categorization problem,

1) The function $\psi: P(X) \to [0, 1]$ is defined by

$$\psi(B) = \max \{\pi(B \cap X_d) \mid d \in D\} \quad \text{for} \quad B \subseteq X,$$

2) If $X = \{X_0, X_1, \ldots, X_p)$

$$X \geq B^{A^c}$$

3) There exists a positive integer $m \geq p + 1$, satisfying:

$$(\forall B \in B^{A^c}) : \psi(B) = 1/m,$$

In order to simplify the analysis we make the following assumptions regarding the state space, cost and information structure of the information-gathering actions.

1) The cost of performing information and communication actions are assumed to be constant as follows:

$$\forall a \in A_1^e, \qquad c(a) = c_1$$

and

$$\forall a \in A_2^e, \qquad c(a) = c_2$$

and

$$\forall a \in A_1^c \cup A_2^c, \qquad c(a) = c$$

2)

$$\forall a \in A_1^e : n_a = \# M_a = k_1$$

and

$$\forall a \in A_2^e : n_a = \# M_a = k_2$$

3) D satisfies

$$(k_1 - 1)\frac{\bar{\omega}}{m} \geq c_1$$

and

$$(k_2 - 1)\frac{\bar{\omega}}{m} \geq c_2 + 2c$$

4)

$$\forall a \in A_1^e, \qquad t(a) = t_1;$$
$$\forall a' \in A_2^e, \quad t(a') = t_2;$$
$$\forall a'' \in A_1^c \cup A_2^c, \qquad t(a'') = t_3;$$

5)

$$X = \{x_0, x_1, \ldots, x_\rho\}$$
$$\phi(x) = \frac{1}{m} = \frac{1}{\rho + 1}$$

Under these conditions, one would like to take as many information-gathering actions as possible, we can prove that the optimal solution in this case is a concurrent strategy (see Appendix 1).

4. Conclusions

In designing DSSs a factor one has to keep in mind is the sequence of information-gathering actions performed by the DSS and the nature of interaction between the system and the human. The sequencing of actions is important when there is a time constraint and the cost of these actions have an impact on the final payoff. These actions may be actually performed by the DSS or may be actions suggested by the DSS for the human to perform on the environment. Within this context one needs to develop analytical techniques for the design for such systems. As a first step in this direction, in this paper we discussed a formal model of decision making which takes into account the fact that problem solving has to be considered from a human–machine perspective. Additionally the cost of information-gathering actions and time is incorporated into the model.

In general the nature of the problem and the characteristics of the problem would determine the type and form of the information-gathering strategy. Here we have considered a special case of the categorization problem called the only correct guesses count problem and within the context of the assumptions made about the problem we have shown that a concurrent information-gathering strategy is optimal. The underlying decision theoretic approach have been applied in areas such as auditing (Fernandez, 1988), networking (Balakrishnan *et al.* 1990) and file search (Moore, Richmond & Whinston, 1988) and expert system development (Hall, Moore & Whinston, 1986).

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Appendix 1

Given the characteristics of the problem discussed in Section 3, we formally discuss the nature of the final partition obtained in an information-gathering strategy and prove that the best strategy is a concurrent strategy. The definitions below present a convenient representation scheme to analyse the problem.

DEFINITION 15

If σ is a feasible strategy, then for each $B \in \boldsymbol{B}_j^s$ where $S \in \{1, 2\}$, $a_s(g, B)$ is defined as follows,

$$a_1(g, B) = \begin{cases} 1 & \text{if } a^1(g, B) \in A_1^e \backslash \{0\} \\ 0 & \text{otherwise} \end{cases}$$

$$a_2(g, B) = \begin{cases} 1 & \text{if } a^2(g, B) \in A_2^e \backslash \{0\} \\ 0 & \text{otherwise} \end{cases}$$

Additionally,

$$a_{1c}(g, B) = \begin{cases} 1 & \text{if } a^1(g, B) \in A_1^c \\ 0 & \text{otherwise} \end{cases}$$

$$a_{2c}(g, B) = \begin{cases} 1 & \text{if } a^2(g, B) \in A_2^c \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 16

Given an efficient information gathering strategy for D,

$$\alpha = \langle [(\boldsymbol{B}_1, \alpha_1^1)], [(\boldsymbol{B}_1^1, \alpha_2^1), (\boldsymbol{B}_2^2, \alpha_2^2)], \ldots, [(\boldsymbol{B}_r^1, \alpha_r^1)], [(\boldsymbol{B}_r^2, \alpha_r^2)] \rangle,$$

τ_j^h, where $h \in \{1, 2, 1c, 2c\}$ is defined as follows:

$$\tau_j^h : \boldsymbol{B}_j^s \rightarrow \{1, 2, \ldots, r+1\} \text{ by}$$

$$\tau_j^h(B) = \sum_{g=1}^{j-1} a_h(g, B), \ B \in \boldsymbol{B}_j^s$$

where $s \in \{1, 2\}$. Additionally,

$$\tau_{\alpha^h}(B) = \tau_{r+1}^h(B), \ B \in \boldsymbol{B}_{r+1}^1$$

PROPOSITION 1

If σ is an efficient strategy for D then (note that B^1_{r+1} is denoted as B_{r+1} as it is assumed that the human makes the final decision).

$$\Gamma(\sigma) = \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)[\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 + (\tau_{\alpha^{1c}}(B) + \tau_{\alpha^{2c}}(B))c]$$

$$= r(c_1 + c_2 + c) - \sum_{B \in B_{r+1}} \pi(B)[r(c_1 + c_2 + c)$$

$$- (\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 + (\tau_{\alpha^{1c}}(B) + \tau_{\alpha^{2c}}(B))c)].$$

Proof

$$\Gamma(\sigma) = \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)C(B)$$

where

$$C(B) = \sum_{j=1}^{r} c[a^1(j, B)] + \sum_{j=1}^{r} c[a^2(j, B)]$$

Since

$$c[a^1(j, B)] = \begin{cases} c_1 & \text{if } a^1(j, B) \in A_1^e \\ c & \text{if } a^1(j, B) \in A_1^c \end{cases}$$

we have

$$\sum_{j=1}^{r} c[a^1(j, B) = \tau_{\alpha^1}(B)c_1 + \tau_{\alpha^{1c}}(B)c$$

Similarly,

$$c[a^2(j, B)] = \begin{cases} c_2 & \text{if } a^2(j, B) \in A_2^e \\ c & \text{if } a^2(j, B) \in A_2^c \end{cases}$$

and

$$\sum_{j=1}^{r} c[a^2(j, B)] = \tau_{\alpha^2}(B)c_2 + \tau_{\alpha^2}(B)c$$

It follows, therefore, that

$$\Gamma(\sigma) = \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)[\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 + (\tau_{\alpha^{1c}}(B) + \tau_{\alpha^{2c}}(B))c]$$

Since

$$\sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)[r(c_1 + c_2 + c)] = r(c_1 + c_2 + c) \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)$$

and

$$\sum_{B \in \boldsymbol{B}_{r+1}} \pi(B) = 1,$$

it also follows that

$$\Gamma(\sigma) = r(c_1 + c_2 + c) - \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)[r(c_1 + c_2 + c)$$

$$- \{(\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 + (\tau_{\alpha^{1c}}(B) + \tau_{\alpha^{2c}}(B))c)\}].$$

We now consider the conditions under which $a \in \Sigma^C$ is optimal is considered.

DEFINITION 17

Let σ be an efficient strategy for D, then we define the following,

$$r_1 = \left\lfloor \frac{T - 2t_3}{t_1} \right\rfloor \quad \text{and}$$

$$r_2 = \left\lfloor \frac{T - 2t_3}{t_2} \right\rfloor \quad \text{and}$$

$$r' = \max \{r_1, r_2\}$$

Lemma 1. If

$$\alpha = \langle [\boldsymbol{B}_1, \alpha_1^1], [\boldsymbol{B}_2^1, \alpha_2^1, (\boldsymbol{B}_2^2, \alpha_2^2], \ldots, [\boldsymbol{B}_r^1, \alpha_r^1], [\boldsymbol{B}_r^2, \alpha_r] \rangle$$

is an efficient and non-trivial concurrent information-gathering strategy for D then

$$\begin{aligned} \#\boldsymbol{B}_{r+1} \leq (k_1^{r_1-2})(k_2^{r_2-2}) - k_2^{r_2-2} \sum_{B \in \boldsymbol{B}_{r'}^1} [k_1^{r_1-2-\tau_{r'}^1(B)} - 1] \\ - k_1^{r_1-2} \sum_{B \in \boldsymbol{B}_{r'}^1} [k_2^{r_2-2-\tau_{r'}^2(B)} - 1] \\ + \sum_{B \in \boldsymbol{B}_{r'}^2} [k_2^{r_2-2-\tau_{r'}^1(B)} - 1] \sum_{B \in \boldsymbol{B}_{r'}^1} [k_1^{r_1-2-\tau_{r'}^1(B)} - 1]. \end{aligned}$$

Proof. Since communication occurs at the first step, the maximum possible number of non-null information gathering steps which can be taken by the human is $r_1 - 2$ and by the computer is $r_2 - 2$ at r'. By Proposition 5.1.6 (Moore & Whinston, (1986), which states that for a single processor $\#B_{r+1} \leq k^r - \sum_{B \in B_{r+1}} [k^{r-\tau_r(B)} - 1]$ we have

$$\#\boldsymbol{B}_{r'}^1 \leq k_1^{r_1-2} \sum_{B \in \boldsymbol{B}_{r'}^1} [k_1^{r_1-2-\tau_{r'}^1(B)} - 1) \tag{1}$$

and

$$\#\boldsymbol{B}_{r'}^2 \leq k_2^{r_2-2} \sum_{B \in \boldsymbol{B}_{r'}^2} [k_2^{r_1-2-\tau_{r'}^2(B)} - 1) \tag{2}$$

Since the computer communicates back to the human at r',

$$\boldsymbol{B}_{r'+1}^1 = \cap \{B_{r'}^1, B_{r'}^2\};$$

and thus

$$\#\boldsymbol{B}_{r'+1}^1 \leq \left[k_1^{r_1-2} - \sum_{B \in \boldsymbol{B}_{r'}^1} [k_1^{r_1-2-\tau_{r'}^1(B)} - 1] \right] \times \left[k_2^{r_2-2} - \sum_{B \in \boldsymbol{B}_{r'}^2} [k_2^{r_2-2-\tau_{r'}^2(B)} - 1] \right].$$

This implies that

$$\begin{aligned} \#\boldsymbol{B}_{r'+1}^1 \leq (k_1^{r_1-2})(k_2^{r_2-2}) - k_2^{r_2-2} \sum_{B \in \boldsymbol{B}_{r'}^1} [k_1^{r_1-2-\tau_{r'}^1(B)} - 1] \\ - k_1^{r_1-2} \sum_{B \in \boldsymbol{B}_{r'}^2} [k_2^{r_2-2-\tau_{r'}^2(B)} - 1] \\ + \sum_{B \in \boldsymbol{B}_{r'}^1} [K_1^{r_1-2-\tau_{r'}^1(B)} - 1] \\ \times \sum_{B \in \boldsymbol{B}_{r'}^2} [k_2^{r_2-2-\tau_{r'}^2(B)} - 1]. \end{aligned} \tag{3}$$

Since $\forall \boldsymbol{B} \in B^1_{r'+1}$, $\alpha^1_j(B) = 0$ and $\forall B \in \boldsymbol{B}^2_{r'+1}$, $\alpha^2_j(B) = 0$ where $j = r' + 1, \ldots, r$ we also have

$$\#\boldsymbol{B}_{r+1} = \#\boldsymbol{B}^1_{r'+1}$$

PROPOSITION 2
If

$$\alpha = \langle [\boldsymbol{B}_1, \alpha^1_1], [\boldsymbol{B}^1_2, \alpha^1_2), (\boldsymbol{B}^2_2, \alpha^2_2)], \ldots, [\boldsymbol{B}^1_r, \alpha^1_r], [\boldsymbol{B}^2_r, \alpha^2_r)]\rangle$$

is an efficient and non trivial concurrent information gathering strategy for D then

$$\begin{aligned}\#B_{r+1} \leq & (k_1^{r_1-2})(k_2^{r_2-2}) \\ & - \sum_{B \in B_{r+1}} [k_1^{r_1-2-\tau_{\alpha^1}(B)} - 1] \\ & - \sum_{B \in B_{r+1}} [k_2^{r_2-2-\tau_{\alpha^2}(B)} - 1] \\ & - \sum_{B \in B_{r+1}} [k_1^{r_1-2-\tau_{\alpha^1}(B)} - 1][k_2^{r_2-2-\tau_{\alpha^2}(B)} - 1] \\ \leq & (k_1^{r_1-2})(k_2^{r_2-2}) \\ & - \sum_{B \in B_{r+1}} [(k_1^{r_1-2-\tau_{\alpha^1}(B)})(k_2^{r_2-2-\tau_{\alpha^2}(B)}) - 1].\end{aligned}$$

Proof.

Let

$$\boldsymbol{B}^1_{r'} = \{B^1_1, \ldots, B^1_{n_1}\} \quad \text{and} \quad B^2_{r'} = \{B^2_1, \ldots, B^2_{n_2}\}.$$

This implies that

$$\boldsymbol{B}^1_{r'+1} = \{\boldsymbol{B}^1_{r'+1}(B^1_1), \ldots, \boldsymbol{B}^1_{r'+1}(B^1_{n_1})\}$$

and

$$\boldsymbol{B}^2_{r'+1} = \{\boldsymbol{B}^2_{r'+1}(B^2_1), \ldots, \boldsymbol{B}^2_{r'+1}(B^2_{n_2})\}$$

where for $B \in \boldsymbol{B}_j$, $\boldsymbol{B}_g(B)$ is defined as the successor of B at $j \in \{1, 2, \ldots, r\}$ and $\boldsymbol{B}_g(B) = \{B' \in \boldsymbol{B}_g \mid B' \cap B = \phi\}$ for $g = j, \ldots, r+1$. Note that the relation holds for both the human and the computer and $\boldsymbol{B}^2_{r+1} = \boldsymbol{B}^2_r$.
For any $B \in B^1_{r'+1}$, there exists a unique $i \in \{1, \ldots, n_1\}$ and $j \in \{1, \ldots, n_2\}$ such that

$$B \in \boldsymbol{B}^1_{r'+1}(B^1_i) \quad \text{and} \quad B \in \boldsymbol{B}^2_{r'+1}(B^2_j)$$

and thus

$$B = B^1_i \cap B^2_j.$$

We see then that, for each $i \in \{1, \ldots, n_1\}$:

$$\boldsymbol{B}^i_{r'+1}(B^1_i) = \{B^1_i \cap B^2_1, \ldots, B^1_i \cap B^2_{n_2}\} \setminus \Phi.$$

Therefore for each $B \in \boldsymbol{B}^1_{r'}$,

$$\#\boldsymbol{B}^1_{r'+1}(B) \leq \#\boldsymbol{B}^2_{r'}; \qquad (1)$$

and since r' is a communication action, for each $B_i \in \boldsymbol{B}_{r'+1}(B)$ we have

$$\tau^1_{r'}(B) = \tau^1_{r'+1}(B_i) = \tau_{\alpha^1}(B_i) \qquad (2)$$

Similarly we have for each $B' \in \boldsymbol{B}^2_{r'}$,

$$\#\boldsymbol{B}^2_{r'+1}(B') \leq \#\boldsymbol{B}^1_{r'}; \tag{3}$$

and for each $B_j \in \boldsymbol{B}^2_{r'+1}(B')$

$$\tau^2_{r'}(B) = \tau^2_{r'+1}(B_j) = \tau_{\alpha^2}(B_j) \tag{4}$$

Now, we have

$$\sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha_1}(B)} - 1] = \sum_{B \in \boldsymbol{B}^1_{r'}} \sum_{B \in \boldsymbol{B}^1_{r'+1}(B_i)} [k_1^{r_1-2-\tau^1_{r'}(B)} - 1];$$

and thus it follows from (1), (2) and Lemma 1 that:

$$\begin{aligned}
\sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha_1}(B)} - 1] &\leq (\#\boldsymbol{B}^2_{r'}) \sum_{B \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau^1_{r'}(B)} - 1] \\
&\leq \left\{ k_2^{r_2-2} - \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau^1_{r'}(B_j)} - 1] \right\} \\
&\quad \times \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_2-2-\tau^1_{r'}(B_i)} - 1] \\
&\leq k_2^{r_2-2} \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau^1_{r'}(B_i)} - 1] \\
&\quad - \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha_2}(B_j)} - 1] \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1].
\end{aligned}$$

This implies

$$\begin{aligned}
\sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha_1}(B)} - 1] + \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha_2}(B_j)} - 1] \times \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1] \\
\leq k_2^{r_2-2} \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1]
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
-k_2^{r_2-2} \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1] \leq - \sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha_1}(B)} - 1] - \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-1-\tau_{\alpha_2}(B_j)} - 1] \\
\times \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1]
\end{aligned} \tag{5}$$

Similarly from (3) and (4) we have

$$\begin{aligned}
\sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_2^{r_2-2-\tau_{\alpha_2}(B)} - 1] \leq k_1^{r_1-2} &\sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha_2}(B_j)} - 1] \\
&- \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1] \\
&\times \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha_2}(B_j)} - 1],
\end{aligned}$$

this inequality can be rewritten as

$$\begin{aligned}
-k_1^{r_1-2} \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha_2}(B_j)} - 1] \leq - &\sum_{B \in \boldsymbol{B}^1_{r'+1}} [k_2^{r_2-2-\tau_{\alpha_2}(B)} - 1] \\
&- \sum_{B_i \in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha_1}(B_i)} - 1] \\
&\times \sum_{B_j \in \boldsymbol{B}^2_{r'}} [k_2^{r'-2-\tau_{\alpha_2}(B_j)} - 1]
\end{aligned} \tag{6}$$

Substituting (5) and (6) into the conclusion of Lemma 1,

$$\begin{aligned}\#\boldsymbol{B}^1_{r'+1} \leq (k_1^{r_1-2})(k_2^{r_2-2}) - \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha 1}(B)}-1] \\ - \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_2^{r_2-2-\tau_{\alpha 2}(B)}-1] \\ - \sum_{B_i\in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau_{\alpha 1}(B_i)}-1] \\ \times \sum_{B_j\in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau_{\alpha 2}(B_j)}-1]\end{aligned} \tag{7}$$

Since $\#\boldsymbol{B}^1_{r'+1} \leq (\#\boldsymbol{B}^1_{r'})\ (\#\boldsymbol{B}^2_{r'})$, and using (2) and (4), the following holds,

$$\sum_{B_i\in \boldsymbol{B}^1_{r'}} [k_1^{r_1-2-\tau^1_{r'}(B_i)}-1] \sum_{B_j\in \boldsymbol{B}^2_{r'}} [k_2^{r_2-2-\tau^2_{r'}(B_j)}-1] \geq \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha 1}(B)}-1][k_2^{r_2-2-\tau_{\alpha 2}(B)}-1] \tag{8}$$

Thus, using (8), (7) can be written as

$$\begin{aligned}\#\boldsymbol{B}^1_{r'+1} &\leq (k_1^{r_1-2})(k_2^{r_2-2}) - \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha 1}(B)}-1] \\ &\quad - \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_2^{r_2-2-\tau_{\alpha 2}(B)}-1] \\ &\quad - \sum_{B\in \boldsymbol{B}^1_{r'+1}} [k_1^{r_1-2-\tau_{\alpha 1}(B)}-1][k_2^{r_2-2-\tau_{\alpha 2}(B)}-1] \\ &\leq (k_1^{r_1-2})(k_2^{r_2-2}) - \sum_{B\in \boldsymbol{B}^1_{r'+1}} \{[k_1^{r_1-2-\tau_{\alpha 1}(B)}-1] \\ &\quad + [k_2^{r_2-2-\tau_{\alpha 2}(B)}-1] \\ &\quad - (k_1^{r_1-2-\tau_{\alpha 1}(B)})(k_2^{r_2-2-\tau_{\alpha 2}(B)}) \\ &\quad - (k_1^{r_1-2-\tau_{\alpha 1}(B)}) - (k_2^{r_2-2-\tau_{\alpha 2}(B)}) + 1\} \\ &\leq (k_1^{r_1-2})(k_2^{r_2-2}) - \sum_{B\in \boldsymbol{B}^1_{r'+1}} \{(k_1^{r_1-2-\tau_{\alpha 1}(B)}) \\ &\quad \times (k_2^{r_2-2-\tau_{\alpha 2}(B)}) - 1\}\end{aligned}$$

PROPOSITION 3

If there exists an efficient concurrent strategy

$$\sigma^* = \langle[(\boldsymbol{B}_1, \alpha_1^1)], [(\boldsymbol{B}^1_2, \alpha^1_2)(\boldsymbol{B}^2_2, \alpha^2_2)], \ldots, [(\boldsymbol{B}^1_r, \alpha^1_r)(\boldsymbol{B}^2_r, \alpha^2_r)], [(\boldsymbol{B}^1_{r+1}, \delta)]\rangle$$

such that,

$$\#\boldsymbol{B}_{r+1} = K_1^{r_1-2}k_2^{r_2-2}$$

then the expected net return from σ^* is

1) $$\Omega^*(\sigma^*) = (k_1^{r_1-2}k_2^{r_2-2})\frac{\bar{\omega}}{m} - [2c + (r_1-2)c_1 + (r_2-2)c_2]$$

and

2) σ^* is optimal for D.

Proof. Corollary 5.1.7 in Moore and Whinston (1986) states that, if

$$\#\boldsymbol{B}_{r+1} > k(k^{r-1} - 1) + 1 \quad \text{then} \quad \forall \boldsymbol{B} \in B_{r+1} : \tau_\alpha(B) = r.$$

Along each path to arrive at r' the number of information gathering actions taken (including null information gathering actions) is $r' - 2$ since the first action taken is the interaction action. However here if

$$\#\boldsymbol{B}^s_{r'} > k_s[k_s^{r_s - 3} - 1] + 1$$

where $s \in \{1, 2\}$ then

$$(\forall B \in \boldsymbol{B}^s_{r'}) : \tau_{\alpha^s}(B) = r_s - 2$$

Consequently since $k^{r-2} > k(k^{r-3} - 1) + 1$, and, since at r' we have an interaction action, hence we have $\forall B \in B_{r'+1}$,

$$\tau_{\alpha^1}(B) = r_1 - 2$$
$$\tau_{\alpha^2}(B) = r_2 - 2$$
$$\tau_{\alpha^{1c}}(B) = 1$$
$$\tau_{\alpha^{2c}}(B) = 1.$$

This implies

$$\Gamma(\sigma^*) = 2c + (r_1 - 2)c_1 + (r_2 - 2)c_2$$

Since

$$\Omega(\sigma^*) = \#\boldsymbol{B}_{r+1} \frac{\bar{\omega}}{m},$$

we see that

$$\Omega^*(\sigma^*) = (k_1^{r_1 - 2})(k_2^{r_2 - 2}) \frac{\bar{\omega}}{m} - [2c + (r_1 - 2)c_1 + (r_2 - 2)c_2]$$

Proof of Optimality

Let σ be any other efficient strategy for D, then by Proposition 2,

$$\#\boldsymbol{B}_{r+1} \leq (k_1^{r_1 - 2})(k_2^{r_2 - 2}) - \sum_{B \in \boldsymbol{B}_{r+1}} [(k_1^{r_1 - 2 - \tau_{\alpha 1}(B)})(k_2^{r_2 - 2 - \tau_{\alpha 2}(B)}) - 1].$$

Letting

$$p = \sum_{B \in \boldsymbol{B}_{r+1}} [(k_1^{r_1 - 2 - \tau_{\alpha 1}(B)})(k_2^{r_2 - 2 - \tau_{\alpha 2}(B)}) - 1] \geq 0,$$

we have,

$$\#\boldsymbol{B}_{r+1} \leq (k_1^{r_1 - 2})(k_2^{r_2 - 2}) - p$$

This implies $\Omega(\sigma) \leq [(k_1^{r_1 - 2})(k_2^{r_2 - 2}) - p] \frac{\bar{\omega}}{m}$, and since by Proposition 1,

$$\Gamma(\sigma^*) = 2c + (r_1 - 2)c_1 + (r_2 - 2)c_2 - \sum_{B \in \boldsymbol{B}_{r+1}} \pi(B)\{[(r_1 - 2)c_1 + (r_2 - 2)c_2$$
$$- \tau_{\alpha^1}(B)c_1 - \tau_{\alpha^2}(B)c_2]\}$$

we have:

$$\Omega(\sigma^*) - \Gamma(\sigma^*) - [\Omega(\sigma) - \Gamma(\sigma)] \geq p\frac{\bar{\omega}}{m} + \sum_{B \in B_{r+1}} \pi(B)[\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 - (r_1 - 2)c_1 - (r_2 - 2)c_2] \quad (1)$$

Define for each $B \in B_{r+1}$,

$$\xi(B) = [(k_1^{r_1-2-\tau_{\alpha 1}(B)})(k_2^{r_2-2-\tau_{\alpha 2}(B)}) - 1]\frac{\bar{\omega}}{m} + \pi(B)[\tau_{\alpha^1}(B)c_1 + \tau_{\alpha^2}(B)c_2 - (r_1 - 2)c_1 - (r_2 - 2)c_2] \quad (2)$$

Letting $B \in B_{r+1}$ be arbitrary, we distinguish three cases:

Case 1. If $\tau_{\alpha^1}(B) = r_1 - 2$ and $\tau_{\alpha^2}(B) = r_2 - 2$ then

$$\xi(B) = [1 - 1]\frac{\bar{\omega}}{m} - \pi(B) \times 0 = 0.$$

Case 2. $\tau_{\alpha^1}(B) = r_1 - 2$ and $\tau_{\alpha^2} \in \{1, 2, \ldots, r' - 3\}$. Here

$$\begin{aligned}\xi(B) &= (k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1)\frac{\bar{\omega}}{m} + \pi(B)[\tau_{\alpha^2}(B)c_2 - (r_2 - 2)c_2] \\ &= (k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1)\frac{\bar{\omega}}{m} - \pi(B)[(r_2 - 2) - \tau_{\alpha_2}(B)]c_2\end{aligned}$$

However, since $k_2 \geq 2$, we have for all $q = 1, 2, \ldots, r_2 - 3$

$$\frac{k_2^q - 1}{q} \geq k_2 - 1; \qquad \frac{[(k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1)]}{r_2 - 2 - \tau_{\alpha^2}(B)}\frac{\bar{\omega}}{m} \geq (k_2 - 1)\frac{\bar{\omega}}{m}; \quad (3)$$

and

$$c_2 \geq \pi(B)c_2. \quad (4)$$

Using (3), (4) and assumption 3 made in the definition of the problem:

$$\xi(B) = (k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1)\frac{\bar{\omega}}{m} - \pi(B)[r_2 - 2) - \tau_{\alpha_2}(B)]c_2 \geq 0$$

Case 3. $\tau_{\alpha^1}(B) \in \{1, 2, \ldots, r_2 - 3\}$ and $\tau_{\alpha^2}(B) = r_2 - 2$. By an argument symmetric with that used in case 2, we can establish that $\xi(B) \geq 0$ here as well.

Case 4. $\tau_{\alpha^1}(B), \tau_{\alpha^2}(B) \in \{1, 2, \ldots, r_2 - 3\}$. It is easy to see that (cf Proposition 2),

$$\begin{aligned}\xi(B) = [k_1^{r_1-2-\tau_{\alpha 1}(B)} - 1)\frac{\bar{\omega}}{m} - \pi(B)[(r_1 - 2) - \tau_{\alpha^1}(B)]c_1 \\ + [k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1]\frac{\bar{\omega}}{m} = \pi(B)[(r_2 - 2) - \tau_{\alpha^2}(B)]c_2 \\ + [k_1^{r_1-2-\tau_{\alpha 1}(B)} - 1][k_2^{r_2-2-\tau_{\alpha 2}(B)} - 1]\frac{\bar{\omega}}{m}\end{aligned} \quad (5)$$

Since we obviously have

$$\left[[k_1^{r_1-2-\tau_{\alpha_1}(B)}-1][k_2^{r_2-2-\tau_{\alpha_2}(B)}-1]\frac{\bar{\omega}}{m}\right]\geq 0$$

and the arguments for case 2 and 3 respectively show that the first and second terms on the right hand side of (5) are non-negative; it follows that

$$\xi(B)\geq 0$$

in this case as well.

Reviewing the four possible cases, we see that for each $B\in B_{r+1}$, we have

$$\xi(B)\geq 0$$

Consequently using (1) we see that

$$\Omega^*(\sigma^*)\geq\Omega^*(\sigma)\geq\sum_{B\in B_{r+1}}\xi(B)\geq 0;$$

and it follows that σ^* is an optimal strategy for the set Σ^C.

In a sequential of mixed strategy with $p>2$ communication points, the maximum number of information gathering steps possible for each information processor is,

$$r_1'\leq\frac{T-pt_3}{t_1} \tag{6}$$

$$r_2'\leq\frac{T-pt_3}{t_2}. \tag{7}$$

From Definition 17, and the definition of a concurrent strategy the maximum number of information-gathering steps possible for a concurrent strategy is r_1+r_2. From (6) and (7) and the definition of a mixed and sequential strategy it is obvious that the maximum number of information-gathering steps possible in these two cases is less than $r_1'+r_2'$. From (1), (2) and Definition 17 we have

$$r_1+r_2>r_1'+r_2' \tag{8}$$

Now, if there exists a concurrent strategy satisfying the condition in the statement of Proposition 3, then as a result of assumption (3) of the problem definition and the inequality specified in (8) above, the concurrent strategy will dominate all strategies in Σ^I, Σ^S and Σ^M.